American University of Beirut
Department of Computer Science
CMPS 211 – Discrete Mathematics – Fall 2014
Assignment 6 Solution

Exercise 1 (10 points)

Let A, B, and C be sets. Show that

a) (A∩B) ⊆ (A∪B∪C).

$\{x|x\epsilon(A\cap B)\}$ ($Assumption$)
=> $\{x|x\epsilon A \wedge x\epsilon B\}$ (Definition of intersection)
=> $\{x|x\epsilon A \vee x\epsilon B\}$
=> $\{x|x\epsilon A \vee x\epsilon B \vee x\epsilon C\}$
=> $\{x|x\epsilon(A\cup B\cup C)\}$

b) (A∩B∩C) ⊆ (A∩B).

$\{x|x\epsilon(A\cap B\cap C)\}$ ($Assumption$)
=> $\{x|x\epsilon A \wedge x\epsilon B \wedge x\epsilon C\}$ (Definition of intersection)
=> $\{x|x\epsilon A \wedge x\epsilon B\}$(Simplification)
=> $\{x|x\epsilon(A\cap B)\}$

c) (A−B)−C ⊆ A−C.

$\{x|x\epsilon(A-B)-C\}(Assumption)$
=> $\{x|x\epsilon A \wedge x \notin B \wedge x \notin C\}$ (Definition of difference)
=> $\{x|x\epsilon A \wedge x \notin C\}$ (Simplification)
=> $\{x|x\epsilon\,(A-C)\}$

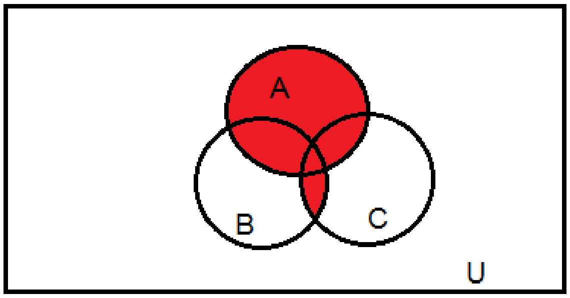
d) (A∩C)∩(B−C) = ∅.

$(A\cap C)\cap(B-C) = (A\cap C)\cap(B\cap\bar{C}) = (A\cap B)\cap(C\cap\bar{C})$
$= (A\cap B)\cap\phi(Complement\ law)$
$= \phi(Domination\ law)$

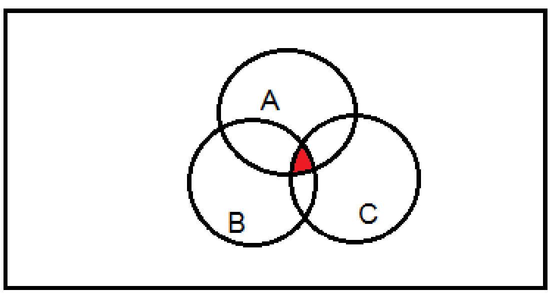
Exercise 2 (10 points)

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

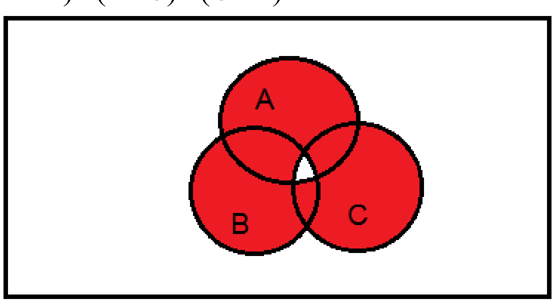
a) A∪ (B∩C)



b) A∩B∩C



c) (A−B)∪(B−C)∪(C−A)



<u>**Exercise 3 (10 points)**</u>

Can you conclude that A = B if A, B, and C are sets such that

a) A∪C = B∪C?
   No,
   Counter example:
   Let A={1,2,3,4,5}, B={1,2,3,4}, C={5}
   AUC = {1,2,3,4,5}
   BUC = {1,2,3,4,5}
   We get, AUC=BUC but A does not equal B.

b) A∩C = B∩C?
   No,
   Counter example:
   Let A={1,2,3}
   B={1,2}
   C={1}
   A∩C={1}= B∩C but A does not equal B.

c) A∪C = B∪C and A∩C = B∩C?

a. Yes,
   Suppose, x ϵ A:
   If x ϵ C, x ϵ A∩C => x ϵ B∩C => x ϵ B
   If x ∉ C, x ϵ AUC => x ϵ BUC => x ϵ B
   - ⇨ A ⊆ B
   - ⇨ Then, do the same starting form x ϵ B until you get B⊆ A
     Therefore, A=B.

d) A−C = B−C and C−A = B−A ?
   No,
   Let A= {1,2,3}, B={1,4,3,6}, C={2,4,6}
   A-C=B-C= {1,3}
   C-A={4,6}= B-A
   But A and B are not equal.

Exercise 4 (10 points)

Find the domain and range of these functions. Note that in each case in order to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns the next smallest integer to a negative integer
Domain: $\mathbb{Z}^-$
Range= $\mathbb{Z}$-{-1}

b) The function that assigns to each nonnegative integer its first digit
Domain= $\mathbb{N}$
Range={0,1,2,3,4,5,6,7,8,9}

c) The function that assigns to a bit string the number of bits in the string
Domain: set of binary strings
Range= $\mathbb{N}$

d) The function that assigns to a bit string the number of zero bits in the string
Domain: set of binary strings
Range= $\mathbb{N}$

Exercise 5 (10 points)

Give an example of a function from **N** to **N** that is

a) one-to-one but not onto.
F(n)=n+2 is one-to-one but not onto since 0 and 1 have no pre-images in $\mathbb{N}$.

b) onto but not one-to-one.
F(n)=$\lfloor n/3 \rfloor$ is onto but not one-to-one since f(0)=f(1)=f(2).

c) both onto and one-to-one (but different from the identity function).
F(n)=$\begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$

d) neither one-to-one nor onto.
F(n)=n! since prime numbers cannot have a pre-image and since f(0)=f(1)=1.

Exercise 6 (10 points)

Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x)=((5x-3)^2-(3x-5)^2)/(4x+4)$
Not a bijection, since there is the condition $x \neq -1$

b) $f(x)= (x+3)/(x+4)$
Not a bijection, since there is the condition $x \neq -4$

c) $f(x)= -4x^2+5$
Not a bijection since neither one-to-one nor onto

d) $f(x)=x^7+3$
Bijection

Exercise 7 (10 points)

Let $f(x) = ax^2 +bx+c$ and $g(x) = dx^2 +ex+f$, where a, b, c, d, e and f are constants. Determine necessary and sufficient conditions on the constants a, b, c, d, e and f so that $f \circ g = g \circ f$.

$f\circ g(x)= f(g(x))= a(cx+d)+b= acx+ad+b$
$g\circ f(x)=g(f(x))=c(ax+b)+d=acx+bc+d$
Then, $f\circ g(x)= g\circ f(x)$ if $ad+b=bc+d$…

Exercise 8 (10 points)

Let f be a function from A to B. Let R and S be subsets of B. Show that

a) $f^{-1}(S\cup R) = f^{-1}(S)\cup f^{-1}(R)$.
Assume $x \in f^{-1}(S)\cup f^{-1}(R)$ then $f(x) \in S$ or $f(x) \in R$.
⇨ $f(x) \in (S\cup R)$
⇨ $x \in F^{-1}(S\cup R)$
⇨ $f^{-1}(S)\cup f^{-1}(R) \subseteq F^{-1}(S\cup R)$

now, assume $x \in F^{-1}(S\cup R)$,then $f(x) \in (S\cup R)$
⇨ $f(x) \in S$ or $f(x) \in R$
⇨ $x \in f^{-1}(S)\cup f^{-1}(R)$
⇨ $F^{-1}(S\cup R) \subseteq f^{-1}(S)\cup f^{-1}(R)$

Therefore, $F^{-1}(S \cup R) = f^{-1}(S) \cup f^{-1}(R)$

b) $f^{-1}(R \cap S) = f^{-1}(S) \cap f^{-1}(R)$.

Assume $\in \boldsymbol{f^{-1}(S) \cap f^{-1}(R)}$ **, then** $\boldsymbol{f(x) \in S \text{ } and \text{ } f(x) \in R}$

⇨ $\boldsymbol{f(x) \in (S \cap R) x}$

⇨ $\boldsymbol{\in f^{-1}(S \cap R)}$

⇨ $\boldsymbol{f^{-1}(S) \cap f^{-1}(R) \subseteq f^{-1}(S \cap R)}$

Let $\boldsymbol{Suppose \text{ } x \in f^{-1}(S \cap R)}$**, then** $\boldsymbol{then \text{ } f(x) \in S \cap R}$

⇨ $\boldsymbol{f(x) \in S \cap R}$

⇨ $\boldsymbol{f(x) \in S \text{ } and \text{ } f(x) \in R}$

⇨ $\boldsymbol{x \in f^{-1}(S) \text{ } and \text{ } x \in f^{-1}(R)}$

⇨ $\boldsymbol{x \in f^{-1}(S) \cap f^{-1}(R)}$

⇨ $\boldsymbol{f^{-1}(S \cap R) \subseteq f^{-1}(S) \cap f^{-1}(R)}$

Therefore, $\boldsymbol{f^{-1}(S \cap R) = f^{-1}(S) \cap f^{-1}(R)}$

Exercise 9 (10 points)

Show that if x is a real number and m is an integer, then $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

$\boldsymbol{Let \text{ } n = \lfloor x \rfloor, where \text{ } n \le x < n + 1}$

⇨ $\boldsymbol{+m \le x + m < n + m + 1}$

⇨ $\boldsymbol{\lfloor x + m \rfloor = n + m = \lfloor x \rfloor + m}$

Exercise 10 (10 points)

Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers x.

True

Proof:

Let n=$\boldsymbol{\lfloor x \rfloor, where \text{ } n \le x < n + 1}$ **such than n**$\in \mathbb{N}$

Then ⌈⌊x⌋⌉=⌈n⌉=n

b) $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

False

Counter example:

Let $x=0.5$ and $y=1.5$

$\lceil x+y \rceil = \lceil 0.5+1.5 \rceil = \lceil 2 \rceil = 2$

$\lceil x \rceil = \lceil 0.5 \rceil = 1$

$\lceil y \rceil = \lceil 1.5 \rceil = 2$

$\lceil x \rceil + \lceil y \rceil = 3$ which is not equal to $\lceil x+y \rceil$

c) $\lceil \lceil x/3 \rceil /2 \rceil = \lceil x/6 \rceil$ all real numbers x.

a. $\lceil \lceil x/3 \rceil /2 \rceil = \lceil x/6 \rceil$

True

Proof:

Let $x = 6n+k$ such that n is an integer and $0 \leq k < 6$

If $k=0$:

We have $x=6n$

⇨ $\lceil \lceil x/3 \rceil /2 \rceil = n = \lceil x/6 \rceil$

If $0<k\leq 3$:

$\lceil x/3 \rceil = \lceil (6n+k)/3 \rceil = \lceil 2n+k/3 \rceil =$ **$2n+1$ (since $0 < k/3 \leq 1$ and 2n is an integer)**

$\lceil \lceil x/3 \rceil /2 \rceil = \lceil (2n+1)/2 \rceil = \lceil n+1/2 \rceil = n+1 = \lceil x/6 \rceil$

If $3<k<6$:

$\lceil x/3 \rceil =$ **$2n+2$**

$\lceil \lceil x/3 \rceil /2 \rceil = n+1 = \lceil x/6 \rceil$

d) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.

False

Counter example:

Let $X=3.1$

$\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{4} \rfloor = 2$

$\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{3.1} \rfloor = \lfloor 1.76 \rfloor = 1$

They are not equal

f) $\lceil x \rceil + \lceil y \rceil + \lceil x+2y \rceil \leq \lceil 2x \rceil + \lceil 3y \rceil$ for all real numbers x and y .

False
Counter example:
Let x=2.1
Y=1.1

⇨ $\lceil x \rceil + \lceil y \rceil + \lceil x+2y \rceil = \lceil 2.1 \rceil + \lceil 1.1 \rceil + \lceil 2.1+2*1.1 \rceil = 3+2+\lceil 4.3 \rceil = 3+2+5=10$
And $\lceil 2x \rceil + \lceil 3y \rceil = \lceil 2*2.1 \rceil + \lceil 3*1.1 \rceil = \lceil 4.2 \rceil + \lceil 3.3 \rceil = 5+4=9 < \lceil x \rceil + \lceil y \rceil + \lceil x+2y \rceil$